Proposition 1. *Let $M = S + A \in \mathbb{R}^{d\times d}$, with S symmetric positive-semi-definite and A antisymmetric, and suppose M is hypocoercive.*

Denote by $\operatorname{Eigvecs}(A)$ *the set of all eigenvectors of A, and* $\operatorname{dist}(U,V) = \inf_{u\in U, z\in V}\|u - z\|$ *for any sets $U, V \subset \mathbb{C}^d$. If $A \neq 0$, then*

$$\min_{\lambda\in\operatorname{Sp}(M)} \Re(\lambda) \geq \left(\frac{1}{\sqrt{\sigma_{\min}(S)}} + \frac{2\sqrt{2\sigma_{\max}(S)}}{\min\limits_{i\mu\neq i\mu'\in\operatorname{Sp}(A)}|\mu-\mu'|}\right)^{-2} \cdot \operatorname{dist}\left(\operatorname{Ker} S, \operatorname{Eigvecs}(A)\cap\mathbb{S}\right)^2.$$

Proof. Let $\lambda \in \operatorname{Sp}(M)$ and $\zeta \in E_\lambda(M)$ such that $\|\zeta\|^2 = 1$. Denote $\lambda = \lambda_1 + i\lambda_2$ ($\lambda_1, \lambda_2 \in \mathbb{R}$). First note that $\overline{\overline{\zeta}^\top S\zeta} = \overline{\zeta}^\top S\zeta \in \mathbb{R}$ and $\overline{\overline{\zeta}^\top A\zeta} = -\overline{\zeta}^\top A\zeta \in i\mathbb{R}$ and so

$$(S + A)\zeta = (\lambda_1 + i\lambda_2)\zeta \quad\Longrightarrow\quad \overline{\zeta}^\top S\zeta = \lambda_1 \text{ and } \overline{\zeta}^\top A\zeta = i\lambda_2.$$

Denote

$$\begin{aligned}\delta &:= A\zeta - (\overline{\zeta}^\top A\zeta)\zeta = A\zeta - i\lambda_2\zeta \\ &= \lambda_1\zeta - S\zeta.\end{aligned}$$

Then,

$$\begin{aligned}\|\delta\|^2 = \|\lambda_1\zeta - S\zeta\|^2 &= \lambda_1^2 - 2\lambda_1\overline{\zeta}^\top S\zeta + \overline{\zeta}^\top S^2\zeta \\ &= -\lambda_1^2 + \overline{\zeta}^\top S^2\zeta \\ &\leq -\lambda_1^2 + \sigma_{\max}(S)\overline{\zeta}^\top S\zeta \\ &= \lambda_1(\sigma_{\max}(S) - \lambda_1) \\ &\leq \sigma_{\max}(S)\lambda_1\end{aligned}$$

(where the first inequality can be checked by decomposing z in the eigenbasis of S). Hence

$$\begin{cases}\lambda_1 \geq \overline{\zeta}^\top S\zeta \\ \lambda_1 \geq \frac{1}{\sigma_{\max}(S)}\|\delta\|^2 = \frac{1}{\sigma_{\max}(S)}\left\|A\zeta - (\overline{\zeta}^\top A\zeta)\zeta\right\|^2\end{cases}$$

$$\text{and so}\quad \min_{\lambda\in\operatorname{Sp}(M)}\Re(\lambda) \geq \inf_{\|z\|=1}\max\left\{\overline{z}^\top Sz,\ \frac{1}{\sigma_{\max}(S)}\left\|Az - (\overline{z}^\top Az)z\right\|^2\right\}.$$

Now fix any $z \in \mathbb{C}^d$ such that $\|z\| = 1$, and denote

$$m = \max\left\{\overline{z}^\top Sz,\ \frac{1}{\sigma_{\max}(S)}\left\|Az - (\overline{z}^\top Az)z\right\|^2\right\}.$$

On one hand, denote the eigenvalue decomposition of S as $S = U\Sigma U^\top$ with $U \in \mathcal{O}_{n+m}(\mathbb{R})$ and Σ diagonal with coefficients $(\sigma_1, ..., \sigma_r, 0, ..., 0)$ where $\sigma_1 \geq ... \geq \sigma_r > 0$. Also let $\Pi = U\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}U^\top$ the (orthogonal) projector onto $\operatorname{Im} S$ along the direction $\operatorname{Ker} S$. Then $S\Pi = \Pi S = S$ and

$$\begin{aligned}m \geq \overline{z}^\top Sz &= \overline{z}^\top\Pi^\top S\Pi z \\ &= \overline{U^\top\Pi z}^\top\Sigma(U^\top\Pi z) \\ &\geq \sigma_{\min}(S)\left\|U^\top\Pi z\right\|^2 = \sigma_{\min}(S)\|\Pi z\|^2 = \sigma_{\min}(S)\operatorname{dist}^2(z, \operatorname{Ker} S)\end{aligned}$$

where $\operatorname{dist}(z, \operatorname{Ker} S) = \inf_{u\in\operatorname{Ker} S}\|z - u\|$.

On the other hand, denote

$$\varepsilon := Az - (\overline{z}^\top Az)z.$$

A is real antisymmetric so normal (meaning it commutes with its adjoint) so it is unitarily diagonalizable. That is, it can be decomposed as $A = V\widetilde{\Sigma}\overline{V}^\top$ where $V \in \mathrm{GL}_{n+m}(\mathbb{C})$ with $V^{-1} = \overline{V}^\top$ and $\widetilde{\Sigma}$ diagonal with coefficients $(i\mu_1, ..., i\mu_{n+m})$ the eigenvalues of A. Then, by Lemma 2 below,

$$\|\varepsilon\| \geq \frac{1}{2\sqrt{2}} \cdot \min_{i\mu \neq i\mu' \in \mathrm{Sp}(A)} |\mu - \mu'| \cdot \mathrm{dist}(z, \mathrm{Eigvecs}(A) \cap \mathbb{S}).$$

Thus we have

$$\sqrt{m} \geq \underbrace{\frac{1}{2\sqrt{2\sigma_{\max}(S)}} \cdot \min_{i\mu \neq i\mu' \in \mathrm{Sp}(A)} |\mu - \mu'|}_{=:1/C} \cdot \mathrm{dist}(z, \mathrm{Eigvecs}(A) \cap \mathbb{S}).$$

Hence we have shown that

$$\sqrt{m} \geq \sqrt{\sigma_{\min}(S)} \cdot \mathrm{dist}(z, \mathrm{Ker}\, S)$$
$$\text{and that} \quad \sqrt{m} \geq \frac{1}{C} \cdot \mathrm{dist}(z, \mathrm{Eigvecs}(A) \cap \mathbb{S}).$$

Letting $u \in \mathrm{Ker}\, S$ such that $\mathrm{dist}(z, \mathrm{Ker}\, S) = \|z - u\|$, as well as $w \in \mathrm{Eigvecs}(A) \cap \mathbb{S}$ such that $\mathrm{dist}(z, \mathrm{Eigvecs}(A) \cap \mathbb{S}) = \|z - w\|$, we have

$$\begin{aligned}\left(\frac{1}{\sqrt{\sigma_{\min}(S)}} + C\right)\sqrt{m} &\geq \|z - u\| + \|z - w\| \\ &\geq \|u - w\| \geq \mathrm{dist}\left(\mathrm{Ker}\, S, \mathrm{Eigvecs}(A) \cap \mathbb{S}\right) \\ m &\geq \left(\frac{1}{\sqrt{\sigma_{\min}(S)}} + C\right)^{-2} \cdot \mathrm{dist}\left(\mathrm{Ker}\, S, \mathrm{Eigvecs}(A) \cap \mathbb{S}\right)^2\end{aligned}$$

as claimed. □

The proof of the proposition used the following technical lemma relating two different ways to measure how far a given unit vector is to being an eigenvector of a normal matrix. It is essentially a quantitative version of the following observation: For any $M \in \mathbb{C}^{d\times d}$ and $z \in \mathbb{C}^d$ such that $\|z\|^2 = 1$, z is an eigenvector of M if and only if $Mz = (\overline{z}^\top Mz)z$, i.e., $(I - z\overline{z}^\top)Mz = 0$.

Lemma 2. *Let $\Sigma \in \mathbb{C}^{d\times d}$ normal, i.e., unitarily diagonalizable. For any $z \in \mathbb{C}^d$ such that $\|z\|^2 = 1$, the vector $\varepsilon = (I - z\overline{z}^\top)\Sigma z$ satisfies*

$$\|\varepsilon\|^2 \geq \frac{1}{8} \min_{\mu \neq \mu' \in \mathrm{Sp}(\Sigma)} |\mu - \mu'|^2 \cdot \mathrm{dist}^2\left(z, \mathrm{Eigvecs}(\Sigma) \cap \mathbb{S}\right).$$

Proof. One can check that it suffices to show the lemma for Σ diagonal. Denote its vector of diagonal coefficients (with repetitions) by $(\mu_1, ..., \mu_d) \in \mathbb{C}^d$, and its set of diagonal coefficients (without repetitions) by $\{\mu_I, I \in \mathcal{I}\} = \mathrm{Sp}(\Sigma)$, and write $[I] = \{i \in [d]; \mu_i = \mu_I\}$. Furthermore, for each I denote by z_I the subvector of z corresponding to indices in $[I]$; by abuse of notation we will also write z_I for the vector in $\mathbb{C}^d$ coinciding with z on indices in $[I]$ and equal to zero elsewhere.

We start by a more explicit estimate for the quantity on the right-hand side of the claimed inequality:

$$\begin{aligned}\mathrm{dist}\left(z, \mathrm{Eigvecs}(\Sigma) \cap \mathbb{S}\right) &= \inf_I \mathrm{dist}\left(z, E_{\mu_I}(\Sigma) \cap \mathbb{S}\right) \\ &= \inf_I \mathrm{dist}\left(z, \mathrm{span}_{\mathbb{C}}\{e_i, i \in [I]\} \cap \mathbb{S}\right) \\ &\leq \inf_I \|z - z_I\| + \mathrm{dist}\left(z_I, \mathrm{span}_{\mathbb{C}}\{e_i, i \in [I]\} \cap \mathbb{S}\right) \\ \mathrm{dist}^2\left(z, \mathrm{Eigvecs}(\Sigma) \cap \mathbb{S}\right) &\leq 2\left(\inf_I \|z - z_I\|^2 + \mathrm{dist}^2\left(z_I, \mathrm{span}_{\mathbb{C}}\{e_i, i \in [I]\} \cap \mathbb{S}\right)\right)\end{aligned}$$

where the first inequality follows by definition of distance to a set and by triangle inequality, and the second from the fact that $(a+b)^2 \leq 2a^2 + 2b^2$ (arithmetic-geometric mean inequality). Now $\|z - z_I\|^2 = \|z\|^2 - \|z_I\|^2 = 1 - \|z_I\|^2$ by definition of z_I as a subvector, and

$$\begin{aligned}\operatorname{dist}^2\left(z_I, \operatorname{span}_{\mathbb{C}}\{e_i, i \in [I]\} \cap \mathbb{S}\right) = \inf_{\substack{\zeta \in \mathbb{C}^{[I]} \\ \|\zeta\|^2 = 1}} \|z_I - \zeta\|^2 &= \|z_I\|^2 + 1 - 2 \sup_{\substack{\zeta \in \mathbb{C}^{[I]} \\ \|\zeta\|^2 = 1}} \Re(\overline{\zeta}^\top z_I) \\ &= \|z_I\|^2 + 1 - 2\|z_I\|.\end{aligned}$$

Thus,

$$\operatorname{dist}^2\left(z, \operatorname{Eigvecs}(\Sigma) \cap \mathbb{S}\right) \leq 4\left(\inf_I 1 - \|z_I\|\right). \tag{1}$$

To compute the quantity on the left-hand side, let $w \in \Delta_{\mathcal{I}}$ the vector with $w_I = \|z_I\|^2 = \sum_{i \in [I]} |z_i|^2$. Then using that Σ is diagonal,

$$\begin{aligned}\|\varepsilon\|^2 = \left\|\Sigma z - (\overline{z}^\top \Sigma z) z\right\|^2 &= \sum_i |z_i|^2 \left|\mu_i - \left(\sum_j \mu_j |z_j|^2\right)\right|^2 \\ &= \mathbb{V}_{i \sim |z_i|^2}(\mu_i) = \mathbb{V}_{I \sim w}(\mu_I)\end{aligned}$$

where $\mathbb{V}_{\omega \sim P}(X(\omega))$ denotes the variance of a (complex) random variable under distribution P. Now, for any distribution P and $X \sim P$, denoting by $\widetilde{X}$ an independent copy of X we have

$$\mathbb{E}\left|X - \widetilde{X}\right|^2 = 2\mathbb{E}|X|^2 - 2|\mathbb{E}X|^2 = 2\mathbb{V}(X).$$

Hence

$$\begin{aligned}\|\varepsilon\|^2 &= \frac{1}{2} \sum_{\substack{I,J \\ I \neq J}} w_I w_J |\mu_I - \mu_J|^2 \\ &\geq \frac{1}{2} \inf_{I \neq J} |\mu_I - \mu_J|^2 \cdot \sum_I \sum_{J \neq I} w_I w_J.\end{aligned} \tag{2}$$

Finally,

$$\begin{aligned}\sum_I \sum_{J \neq I} w_I w_J &= \sum_I w_I (1 - w_I) \\ &\geq \inf_I (1 - w_I) = \inf_I (1 - \|z_I\|^2) \geq \inf_I (1 - \|z_I\|)\end{aligned}$$

since $w \in \Delta_{\mathcal{I}}$ and $\|z_I\| \leq 1$ for each I. So, putting together (1) and (2), we get

$$\begin{aligned}\|\varepsilon\|^2 &\geq \frac{1}{2} \inf_{I \neq J} |\mu_I - \mu_J|^2 \cdot \inf_I (1 - \|z_I\|) \\ &\geq \frac{1}{2} \inf_{I \neq J} |\mu_I - \mu_J|^2 \cdot \frac{1}{4} \operatorname{dist}^2\left(z, \operatorname{Eigvecs}(\Sigma) \cap \mathbb{S}\right).\end{aligned}$$

as announced. □